1.3.2 Limit point compactness, sequential compactness

X is limit point compact if every infinite subset of X has a limit point.
X is sequentially compact if every sequence of points in X has a convergent subsequence.
Compact $\Rightarrow$ Countably compact $\Rightarrow$ Limit point compact.
Sequentially compact $\Rightarrow$ Countably compact

Example:
Let Y be a 2-point indiscrete space. Then $\mathbb{Z}_+ \times Y$ is limit point compact since every nonempty subset has a limit point, but it is not compact since the covering by the open sets $\{n\} \times Y$ has no finite subcover.

In a metrizable space, compactness, countable compactness, limit point compactness and sequential compactness are equivalent.
In a second countable Hausdorff space, compactness, countable compactness, limit point compactness and sequential compactness are equivalent.
If nets are used instead of sequences, then compactness and sequential compactness are equivalent.
In a T_1 space, countable compactness and limit point compactness are equivalent.

1.3.3 Local compactness

X is locally compact if for each $x \in X$ there is a compact subspace of X containing a neighborhood of x.

Every linearly ordered set with the least upper bound property is locally compact: Every basis element is contained in a closed interval.
$\mathbb{R}_l$ is not locally compact because its compact subsets must be countable.

X is locally compact Hausdorff if and only if there exists a space Y such that (i) X is a subspace of Y, (ii) $Y \setminus X$ consists of a single point, (iii) Y is a compact Hausdorff space.
If Y, Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that is the identity on X.
The point in $Y \setminus X$ is denoted by ∞. The topology on Y consists of (i) all open sets in X, (ii) all sets of the form $Y \setminus C$ where C is a compact subspace of X.
If X is not compact, then ∞ is a limit point of X so $\bar{X} = Y$.
Y is called the one-point compactification of X.

If X is Hausdorff, then X is locally compact if and only if for every $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.

A subspace of a locally compact Hausdorff space that is open or closed is locally compact.
X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact.
A proper map between locally compact Hausdorff spaces extends to a continuous map between their one-point compactifications. It follows that such maps are closed.

Local compactness is preserved under finite products.
For infinite products, the product space is locally compact if and only if each factor is locally compact and all but finitely many of them are compact.
Local compactness is preserved under open continuous maps and perfect maps.

1.4 Countability Axioms

1.4.1 First countability

X has a countable basis at x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains some element of $\mathcal{B}$.
X is first countable if it has a countable basis at each point.

If a sequence of points of A converges to x, then $x \in \bar{A}$.